

Edge-Magic Total Labeling on Vertex Amalgamation Graphs of a Star Graph with a Path Graph

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ABSTRACT

One of the topic graph theories is graph labeling. Let G be a finite simple connected graph, a bijection f from $V(G) \cup E(G)$ to $\{1, 2, \dots, p + q\}$ where $p = |V(G)|$ and $q = |E(G)|$ is called an edge-magic total labeling of G if there exists a constant k (called the magic sum of f) such that $f(u) + f(v) + f(uv) = k$ for any edge uv of G . The super edge-magic total labeling on a graph G is the edge-magic total labeling which maps V into the set $\{1, 2, \dots, p\}$. Let G_i be a connected graph with a fixed vertex $v_{oi} \in V(G_i)$. The vertex amalgamation of graph G_i onto a fixed vertex v_{oi} called terminal denoted by $\text{Amal}(G_i, v_{oi})$ is a graph formed by taking all elements (vertices and edges) in G_i with $v_{oi} = v_{oj}, \forall i, j$. In this study, we will show that vertex amalgamation graphs of a star graph with a path graph are edge-magic total and super edge-magic total labeling, with constructed vertex labelings and edge labelings to obtain intervals of the magic sums k .

Keywords: Graph Theory, Broom Graphs, Graph Labeling

INTRODUCTION

The development of graph theory has been due in big measure to its essential role in the applied sciences (Aytaç, 2020). The link between graph theory and other branches of mathematics are becoming progressively strong (Mallik & Ghosh, 2018).

According to historical records, graph theory originated from the solution to the Königsberg bridge problem introduced by Leonhard Euler, a famous Swiss mathematician in 1736, in his writings entitled "Solutio Problematis and Geometriae Pertinentis Sites". Thanks to Euler's work, which was inspired through the Königsberg bridge problem, a fairly important branch of mathematics was created known as graph theory (Mondal & De, 2017).

In forming a new graph, one way that can be done is by using the amalgamation operation (Chatterjee et al., 2020). In this study, the authors will use vertex amalgamation operations. A vertex amalgamation of pairs of graph vertices (G, u) together is the resulting graph (H, v) by joining the vertices and becoming one fixed vertex uv (Abu-Khzam et al., 2018).

In the development of graph theory there are several fields of study, one of which is graph labeling (Chartrand et al., 2019). Graph labeling is a topic of combinatoric mathematics research that has developed rapidly in recent years. An edge-magic total labeling is one of the most important types of labeling (López et al., 2017; Mohan, 2019). A graph that fulfills edge-magic total labeling is a graph that is labeled with a positive integer, where each vertex and the associated edge (incident) when added together produce the same positive integer (Chartrand et al., 2019). The edge-magic total labeling is called super edge-magic total labeling if each vertex is labeled with the smallest positive integer (Bača et al., 2019; Farahmand Asil, 2018; Maowa, 2016; Marimuthu &

Kumar, 2015; Slater, 2016; Yao & Wang, 2021).

As far as we know, for the family of graphs $\text{Amal}(S_n; P_4, v_{oi})$, there have not been any traditional manual method that can obtain its edge-magic total and super edge-magic total labeling with intervals of the magic sums. Now, we design a new edge-magic total and super edge-magic total labeling manual method with intervals of magic sums for the family of graphs $\text{Amal}(S_n; P_4, v_{ok})$ based on the graph labeling method.

RESEARCH METHODE

This research uses a literature review method to collect, evaluate, and synthesize relevant information from related literature. This approach allows us to understand the development of graph theory from a variety of sources, including the contributions of leading figures in the field throughout history. Any sector of mathematics about any scientific activity, we feel that appreciation of graph theory is enhanced by being familiar with most of the people, present and in the past, who were or are responsible for its growth (Chartrand et al., 2019). General graph theory principles are used in various fields for research and different application models. This includes studying biodiversity, conservation and even to quantify actor popularity or to investigate diffusion processes (Andriollo et al., 2023).

Definition 1. A graph G consists of a finite nonempty set V of objects called vertices (a vertex for the singular) and a set E of 2-element subsets of V called edges. The sets V is vertex set and the sets E is edge set of G . Thus, a graph G is an ordered pair of two sets V and E [9].

Thus, the literature review method in this research plays a key role in presenting a comprehensive understanding of the development of graph theory and its broad applications in various fields of science.

A. Types of Graphs

In this study, we provide two types of broom graphs that will be used in this study.

Definitions 2. A broom graph $SP_{n,m}$ is a connected graph of order $(n + m)$ which has $(m - 1)$ vertices of degree 1, n vertices of degree 2, and 1 vertex of degree $(m - 1)$. A broom graph can be constructed through a vertex amalgamation between a path graph and a star graph [1].

Definitions 3. A broom graph can also be defined through an operation of vertex amalgamation between a path graph and a star graph. Suppose $P_n: v_1, e_1, \dots, v_{n-1}, e_{n-1}, v_n$ is a path graph of order n and S_m is a star graph of order m at a fixed vertex u_1 .

A broom graph of order $n + m - 1$ denoted $SP_{n,m}$ is $\text{Amal}(P_n; S_m, v_1; u_1)$ is a graph that consists of all vertices and edges of P_n and S_m where vertices $v_1 = u_1$ [1].

B. The Vertex Amalgamation

In this section, we present a definition of a vertex amalgamation of graph.

Definition 4. Let G_i graph is connected with a fixed vertex $v_{oi} \in V(G_i)$. The amalgamation of graph G_i at a fixed vertex called a terminal denoted by $\text{Amal}(G_i, v_{oi})$ is a graph formed by taking all the elements (vertices and edges) of G_i with $v_{oi} = v_{oj}, \forall i$. The graph $\text{Amal}(G_i, v_{oi})$ can also be written as $\text{Amal}(G_1; G_2; \dots; G_k, v_{o1}; v_{o2}; \dots v_{ok})$. If $G_i = G_j$ for each i, j , then $\text{Amal}(G_1; G_2; \dots; G_k, v_{o1}; v_{o2}; \dots v_{ok})$ can be written as $\text{Amal}(kG_i, v_{oi})$ for all $i \in$

$\{1, 2, \dots, k\}$ [1].

C. Edge-Magic Total Labeling and Super Edge-Magic Total Labeling

In this section, we present some definitions of edge-magic total and super edge-magic total labeling.

Definition 5. Let $G(p, q)$ be a finite simple connected graph, a bijection f from $V(G) \cup E(G)$ to $\{1, 2, \dots, p + q\}$ is called an edge-magic total labeling (EMTL for short) of G if there exists a constant k (called the magic sum of f) such that $f(u) + f(uv) + f(v) = k$ for any edge uv of G [11].

Definition 6. An edge-magic total labeling f of a (p, q) graph G is called super edge-magic total (super EMTL) if $f(V(G)) = \{1, 2, \dots, p\}$ [10].

According to the definition of super edge-magic total labeling, we know that the sum of all labels assigned to vertices and edges of a graph $G(p, q)$ is denoted by C , the formula

$$C = L_p + L_q = \frac{(p + q)(p + q + 1)}{2}, \quad (2.1)$$

Where L_p and L_q stand for the sum of labels assigned to the vertices and the sum of labels assigned to the edges, respectively. On the other hand, $\deg(v_i)$ is the degree of vertex v_i [11], we get the formula (2.2).

$$k = \frac{C + \sum_{i=1}^p (\deg(v_i) - 1)f(v_i)}{q}.$$

RESULTS AND DISCUSSION

A. Vertex Amalgamations of a Star Graph with a Path Graph P_4

The following definition of a vertex amalgamation of graph is taken from [1]. Suppose S_n and P_4 are subgraphs of a graph $G = (V, E)$ and $v_{0k} \in V(G)$. The graph G is called the vertex amalgamation of S_n and P_4 at a fixed vertex v_{0k} , denoted by $G = Amal(S_n; P_4, v_{0k})$, if $G = S_n \cup P_4$ and $S_n \cap P_4 = \{v_{0k}\}$. So, the amalgamation $Amal(S_n; P_4, v_{0k})$ is a graph formed by taking all the elements (vertices and edges) of G .

B. Intervals of Magic Sums on Vertex Amalgamation Graphs of a Star Graph with a Path Graph P_4

In this section, we present intervals of magic sum of four new graphs on vertex amalgamation graphs of a star graph S_n with a path graph P_4 .

a. Magic Sums of a Graph $Amal(S_n; P_4, x_2)$

Suppose $Amal(S_n; P_4, x_2)$ obtains a set of vertices and a set of edges

$$V(Amal(S_n; P_4, x_2)) = \{x_1, x_2, x_3, x_4, x_5\} \cup \{y_j | 1 \leq j \leq n - 2\} \text{ and}$$

$$E(Amal(S_n; P_4, x_2)) = \{x_1x_2, x_2x_3, x_2x_4, x_4x_5\} \cup \{x_1y_j | 1 \leq j \leq n - 2\}.$$

Suppose C is the sum of all labels assigned to vertices and edges of a graph $Amal(S_n; P_4, x_2)$, thus the vertex sets and edge sets are labeled from number 1 to $(p + q)$. Furthermore, by substituting the value $V(Amal(S_n; P_4, x_2)) = p = n + 3$ and $E(Amal(S_n; P_4, x_2)) = q = n + 2$ into Eq. (2.1) we get

$$C = \frac{(p + q)(p + q + 1)}{2} \\ C = 2n^2 + 11n + 15. \quad (3.1)$$

Theorem 3.1. Let $Amal(S_n; P_4, x_2)$ is a graph of a vertex amalgamation of a star graph with a path graph P_4 at a fixed vertex x_2 and k is the magic sums of $Amal(S_n; P_4, x_2)$, then the magic sums are in closed interval

$$\frac{2n^2 + 12n + 20}{n + 2} \leq k \leq \frac{4n^2 + 18n + 16}{n + 2}.$$

Proof. Based on Eq. (2.2), the magic sums k have a minimum number when the vertices with the highest degree are labeled with the smallest labels. Furthermore, by substituting Eq. (3.1) to Eq. (2.2) we get

$$\begin{aligned} k &= \frac{C + \sum_{i=1}^p (deg(v_i) - 1)f(v_i)}{q} \\ &= \frac{(2n^2 + 11n + 15) + (2 - 1) \cdot f(x_4) + (3 - 1) \cdot f(x_2) + (n - 1 - 1) \cdot f(x_1)}{n + 2} \\ &= \frac{(2n^2 + 11n + 15) + 1 \cdot 3 + 2 \cdot 2 + (n - 2) \cdot 1}{n + 2} \\ &= \frac{2n^2 + 12n + 20}{n + 2}. \end{aligned}$$

Based on Eq. (2.2), the magic sums k have a minimum number when the vertices with the highest degree are labeled with the highest labels. Furthermore, by substituting Eq. (3.1) to Eq. (2.2) we get

$$\begin{aligned} k &= \frac{C + \sum_{i=1}^p (deg(v_i) - 1)f(v_i)}{q} \\ &= \frac{(2n^2 + 11n + 15) + (2 - 1) \cdot f(x_4) + (3 - 1) \cdot f(x_2) + (n - 1 - 1) \cdot f(x_1)}{n + 2} \\ &= \frac{(2n^2 + 11n + 15) + 1 \cdot (2n + 3) + 2 \cdot (2n + 4) + (n - 2) \cdot (2n + 5)}{n + 2} \\ &= \frac{4n^2 + 18n + 16}{n + 2}. \end{aligned}$$

Therefore, it is proved that $Amal(S_n; P_4, x_2)$ has magic sums in closed interval

$$\frac{2n^2 + 12n + 20}{n + 2} \leq k \leq \frac{4n^2 + 18n + 16}{n + 2}.$$

b. Magic Sums of Broom Graph $SP_{3,n}$

Suppose $SP_{3,n}$ obtains a set of vertices and a set of edges

$$V(SP_{3,n}) = \{x_1, x_2, x_3, x_4, x_5\} \cup \{y_j | 1 \leq j \leq n - 2\} \text{ and}$$

$$E(SP_{3,n}) = \{x_1x_2, x_2x_3, x_3x_4, x_4x_5\} \cup \{x_1y_j | 1 \leq j \leq n - 2\}.$$

Suppose C is the sum of all labels assigned to vertices and edges of a graph $SP_{3,n}$, thus the vertex sets and edge sets are labeled from number 1 to $(p + q)$. Furthermore, by substituting the value $V(SP_{3,n}) = p = n + 3$ and $E(SP_{3,n}) = q = n + 2$ into Eq. (2.1) we get

$$\begin{aligned} C &= \frac{(p + q)(p + q + 1)}{2} \\ C &= 2n^2 + 11n + 15. \end{aligned} \tag{3.2}$$

Theorem 3.2. Let $SP_{3,n}$ is a vertex amalgamation of a star graph with a path graph P_4 and k is the magic sums of $SP_{3,n}$, then magic sums are in closed interval

$$\frac{2n^2 + 12n + 22}{n + 2} \leq k \leq \frac{4n^2 + 18n + 14}{n + 2}.$$

Proof. Based on Eq. (2.2), the magic sums k have a minimum number when the vertices with the highest degree are labeled with the smallest labels. Furthermore, by substituting Eq. (3.2) to Eq. (2.2) we get

$$\begin{aligned} k &= \frac{C + \sum_{i=1}^p (\text{deg}(v_i) - 1)f(v_i)}{q} \\ &= \frac{(2n^2 + 11n + 15) + (2 - 1)(f(x_2) + f(x_2) + f(x_3)) + (n - 1 - 1) \cdot f(x_1)}{n + 2} \\ &= \frac{(2n^2 + 11n + 15) + 1(2 + 3 + 4) + (n - 2) \cdot 1}{n + 2} \\ &= \frac{2n^2 + 12n + 22}{n + 2}. \end{aligned}$$

Based on Eq. (2.2), the magic sums k have a minimum number when the vertices with the highest degree are labeled with the highest labels. Furthermore, by substituting Eq. (3.2) to Eq. (2.2) we get

$$\begin{aligned} k &= \frac{C + \sum_{i=1}^p (\text{deg}(v_i) - 1)f(v_i)}{q} \\ &= \frac{(2n^2 + 11n + 15) + (2 - 1)(f(x_2) + f(x_2) + f(x_3)) + (n - 1 - 1) \cdot f(x_1)}{n + 2} \\ &= \frac{(2n^2 + 11n + 15) + 1(2n + 2 + 2n + 3 + 2n + 4) + (n - 2) \cdot (2n + 5)}{n + 2} \\ &= \frac{4n^2 + 18n + 14}{n + 2}. \end{aligned}$$

Therefore, it is proved that $SP_{3,n}$ has magic sums in closed interval

$$\frac{2n^2 + 12n + 22}{n + 2} \leq k \leq \frac{4n^2 + 18n + 14}{n + 2}.$$

c. Magic Sums of a Broom Graph $SP_{4,n}$

Suppose $SP_{4,n}$ obtains a set of vertices and a set of edges

$$\begin{aligned} V(SP_{4,n}) &= \{x_1, x_2, x_3, x_4\} \cup \{y_j | 1 \leq j \leq n - 1\} \text{ and} \\ E(SP_{4,n}) &= \{x_1x_2, x_2x_3, x_3x_4\} \cup \{x_1y_j | 1 \leq j \leq n - 1\}. \end{aligned}$$

Suppose C is the sum of all labels assigned to vertices and edges of a graph $SP_{4,n}$, thus the vertex sets and edge sets are labeled from number 1 to $(p + q)$. Furthermore, by substituting the value $V(SP_{4,n}) = p = n + 3$ and $E(SP_{4,n}) = q = n + 2$ into Eq. (2.1) we get

$$\begin{aligned} C &= \frac{(p + q)(p + q + 1)}{2} \\ C &= 2n^2 + 11n + 15. \end{aligned} \tag{3.3}$$

Theorem 3.3. Let $SP_{4,n}$ is a vertex amalgamation of a star graph with a path graph P_4 and k is the magic sums of $SP_{3,n}$, then magic sums are in closed interval

$$\frac{2n^2 + 12n + 19}{n + 2} \leq k \leq \frac{4n^2 + 18n + 17}{n + 2}.$$

Proof. Based on Eq. (2.2), the magic sums k have a minimum number when the vertices with the highest degrees are labeled with the smallest labels. Furthermore, by substituting Eq. (3.3) to Eq. (2.2) we get

$$\begin{aligned}
 k &= \frac{C + \sum_{i=1}^p (\text{deg}(v_i) - 1)f(v_i)}{q} \\
 &= \frac{(2n^2 + 11n + 15) + (2 - 1) \cdot f(x_2) + (2 - 1) \cdot f(x_3) + (n - 1) \cdot f(x_1)}{n + 2} \\
 &= \frac{(2n^2 + 11n + 15) + 1 \cdot 3 + 1 \cdot 2 + (n - 1) \cdot 1}{n + 2} \\
 &= \frac{2n^2 + 12n + 19}{n + 2}.
 \end{aligned}$$

Furthermore, Based on Eq. (2.2), the magic sums k have a minimum number when the vertices with the highest degrees are labeled with the highest labels. Furthermore, by substituting Eq. (3.3) to Eq. (2.2) we get

$$\begin{aligned}
 k &= \frac{C + \sum_{i=1}^p (\text{deg}(v_i) - 1)f(v_i)}{q} \\
 &= \frac{(2n^2 + 11n + 15) + (2 - 1) \cdot f(x_2) + (2 - 1) \cdot f(x_3) + (n - 1) \cdot f(x_1)}{n + 2} \\
 &= \frac{(2n^2 + 11n + 15) + 1 \cdot (2n + 3) + 1 \cdot (2n + 4) + (n - 1) \cdot (2n + 5)}{n + 2} \\
 &= \frac{(2n^2 + 11n + 15) + 4n + 7 + 2n^2 + 3n - 5}{n + 2} \\
 &= \frac{4n^2 + 18n + 17}{n + 2}.
 \end{aligned}$$

Therefore, it is proved that $SP_{4,n}$ has magic sums in closed interval

$$\frac{2n^2 + 12n + 19}{n + 2} \leq k \leq \frac{4n^2 + 18n + 17}{n + 2}.$$

d. Magic Sums of $Amal(S_n; P_4, x_1)$

Suppose $Amal(S_n; P_4, x_1)$ obtains a set of vertices and a set of edges

$$V(Amal(S_n; P_4, x_1)) = \{x_1, x_2, x_3\} \cup \{y_j | 1 \leq j \leq n\} \text{ and}$$

$$E(Amal(S_n; P_4, x_1)) = \{x_1x_2, x_2x_3\} \cup \{x_1y_j | 1 \leq j \leq n\}.$$

Suppose C is the sum of all labels assigned to vertices and edges of a graph $Amal(S_n; P_4, x_1)$, thus the vertex sets and edge sets are labeled from number 1 to $(p + q)$. Furthermore, by substituting the value $V(Amal(S_n; P_4, x_1)) = p = n + 3$ and $E(Amal(S_n; P_4, x_1)) = q = n + 2$ into Eq. (2.1) we get

$$\begin{aligned}
 C &= \frac{(p + q)(p + q + 1)}{2} \\
 C &= 2n^2 + 11n + 15.
 \end{aligned} \tag{3.4}$$

Theorem 3.4. Let $Amal(S_n; P_4, x_1)$ is a vertex amalgamation of a star graph with a path graph P_4 at a fixed vertex x_2 and k is the magic sums of $Amal(S_n; P_4, x_1)$, then the magic sums are in closed interval

$$\frac{2n^2 + 12n + 17}{n + 2} \leq k \leq \frac{4n^2 + 18n + 19}{n + 2}.$$

Proof. Based on Eq. (2.2), the magic sums k have a minimum number when the vertices with the highest degree are labeled with the smallest labels. Furthermore, by substituting Eq. (3.4) to Eq. (2.2) we get

$$k = \frac{C + \sum_{i=1}^p (\text{deg}(v_i) - 1)f(v_i)}{q}$$

$$\begin{aligned}
&= \frac{(2n^2 + 11n + 15) + (2 - 1) \cdot f(x_2) + (n + 1 - 1) \cdot f(x_1)}{n + 2} \\
&= \frac{(2n^2 + 11n + 15) + 1 \cdot 2 + (n) \cdot 1}{n + 2} \\
&= \frac{2n^2 + 12n + 17}{n + 2}.
\end{aligned}$$

Based on Eq. (2.2), the magic sums k have a minimum number when the vertices with the highest degree are labeled with the highest labels. Furthermore, by substituting Eq. (3.4) to Eq. (2.2) we get

$$\begin{aligned}
k &= \frac{C + \sum_{i=1}^p (\deg(v_i) - 1)f(v_i)}{q} \\
&= \frac{(2n^2 + 11n + 15) + (2 - 1) \cdot f(x_2) + (n + 1 - 1) \cdot f(x_1)}{n + 2} \\
&= \frac{(2n^2 + 11n + 15) + 1 \cdot (2n + 4) + (n) \cdot (2n + 5)}{n + 2} \\
&= \frac{4n^2 + 18n + 19}{n + 2}.
\end{aligned}$$

Therefore, it is proved that $Amal(S_n; P_4, x_1)$ has magic sums in closed interval

$$\frac{2n^2 + 12n + 17}{n + 2} \leq k \leq \frac{4n^2 + 18n + 19}{n + 2}.$$

3.3. Edge-Magic Total Labelings on Vertex Amalgamation Graphs of a Star Graph with a Path Graph P_4

In this section, the study will discuss about edge-magic total and super edge-magic total labelings on vertex amalgamation graphs of star graphs with path graphs P_4 .

a. Edges-Magic Total Labelings of a graph $Amal(S_n; P_4, x_2)$

There are two types of the edges-magic total labeling on a graph $Amal(S_n; P_4, x_2)$ as follows:

i. Edge-Magic Total Labeling of a Graph $Amal(S_n; P_4, x_2)$

Theorem 3.5. If $n \geq 3$, then the graph $Amal(S_n; P_4, x_2)$ has an edge-magic total labeling with $k = 4n + 6$.

Proof. Suppose $Amal(S_n; P_4, x_2)$ obtains a set of vertices and a set of edges

$$V(Amal(S_n; P_4, x_2)) = \{x_1, x_2, x_3, x_4, x_5\} \cup \{y_j | 1 \leq j \leq n - 2\} \text{ and}$$

$$E(Amal(S_n; P_4, x_2)) = \{x_1x_2, x_2x_3, x_2x_4, x_4x_5\} \cup \{x_1y_j | 1 \leq j \leq n - 2\}.$$

The cardinality of the vertex set and edge set are respectively $|V(Amal(S_n; P_4, x_2))| = p = n + 3$ and $|E(Amal(S_n; P_4, x_2))| = q = n + 2$. Define a bijection f from $V(Amal(S_n; P_4, x_2)) \cup E(Amal(S_n; P_4, x_2))$ to $\{1, 2, \dots, p + q\}$ with the labelings as follows.

$$\begin{aligned}
&f(x_1) = 1; f(x_2) = n + 2; f(x_3) = 2, f(x_4) = 3; f(x_5) = n + 3; f(y_j) = j + 3; \\
&f(x_1x_2) = n + 7; f(x_2x_3) = n + 6; f(x_2x_4) = n + 5; f(x_4x_5) = n + 4 \text{ and} \\
&f(x_1y_j) = 2n - j + 6, \text{ for all } 1 \leq j \leq n - 2.
\end{aligned}$$

Because each edge holds $f(u) + f(uv) + f(v) = k$. So, it is proved that for $n \geq 3$, then the graph $Amal(S_n; P_4, x_2)$ has an edge-magic total labeling with $k = 4n + 6$.

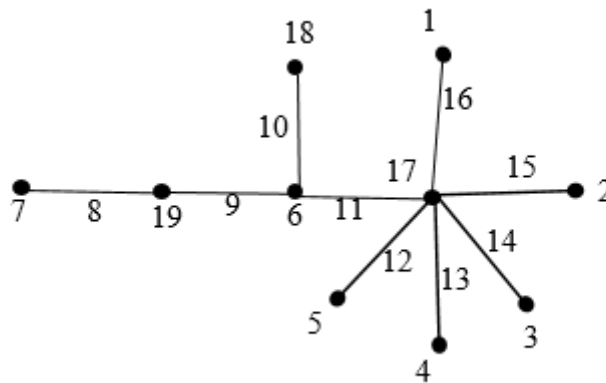


Figure 1. An EMTL of $Amal(S_7;P_4,x_2)$ with the magic sum $k=34$.

ii. Super Edge-Magic Labeling of a graph $Amal(S_n;P_4,x_2)$

Theorem 3.6. If $n \geq 2$, then the graph $Amal(S_n;P_4,x_2)$ has an edge-magic total labeling with $k = 2n + 10$.

Proof. Suppose $Amal(S_n;P_4,x_2)$ obtains a set of vertices and a set of edges

$$V(Amal(S_n;P_4,x_2)) = \{x_1, x_2, x_3, x_4, x_5\} \cup \{y_j | 1 \leq j \leq n - 2\} \text{ and}$$

$$E(Amal(S_n;P_4,x_2)) = \{x_1x_2, x_2x_3, x_2x_4, x_4x_5\} \cup \{x_1y_j | 1 \leq j \leq n - 2\}.$$

The cardinality of the vertex set and edge set are respectively $|V(Amal(S_n;P_4,x_2))| = p = n + 3$ and $|E(Amal(S_n;P_4,x_2))| = q = n + 2$. Define a bijection f from $V(Amal(S_n;P_4,x_2)) \cup E(Amal(S_n;P_4,x_2))$ to $\{1, 2, \dots, p + q\}$ with the labeling as follows.

$$f(x_1) = 1; f(x_2) = n + 2; f(x_3) = 2; f(x_4) = 3; f(x_5) = n + 3; f(y_j) = j + 3;$$

$$f(x_1x_2) = n + 7; f(x_2x_3) = n + 6; f(x_2x_4) = n + 5; f(x_4x_5) = n + 4 \text{ and}$$

$$f(x_1y_j) = 2n - j + 6, \text{ for all } 1 \leq j \leq n - 2.$$

Because $f(V(G)) = \{1, 2, \dots, p\}$ and each edge holds $f(u) + f(uv) + f(v) = k$. So, it is proved that for $n \geq 2$, then the graph $Amal(S_n;P_4,x_2)$ has a super edge-magic total labeling with $k = 2n + 10$.

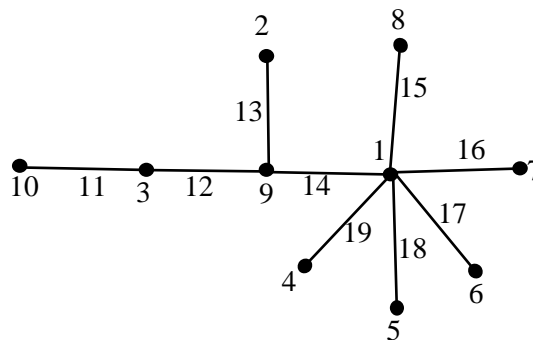


Figure 2. A super EMTL of $Amal(S_7;P_4,x_2)$ with the magic sum $k = 24$.

b. Edges-Magic Total Labelings of a Broom $SP_{3,n}$

There are two types of the edges-magic total labeling on a broom graph $SP_{3,n}$ as follows:

1. Edge-Magic Total Labeling of a Broom Graph $SP_{3,n}$

Theorem 7: If $n \geq 2$, then the broom graph $SP_{3,n}$ has an edge-magic total labeling with $k = 2n + 10$.

Proof. Suppose $SP_{3,n}$ obtains a set of vertices and a set of edges

$$V(SP_{3,n}) = \{x_1, x_2, x_3, x_4, x_5\} \cup \{y_j | 1 \leq j \leq n - 2\} \text{ and}$$

$$E(SP_{3,n}) = \{x_1x_2, x_2x_3, x_3x_4, x_4x_5\} \cup \{x_1y_j | 1 \leq j \leq n - 2\}.$$

The cardinality of the vertex set and edge set are respectively $|V(SP_{3,n})| = p = n + 3$ and $|E(SP_{3,n})| = q = n + 2$. Define a bijection f from $V(SP_{3,n}) \cup E(SP_{3,n})$ to $\{1, 2, \dots, p + q\}$ with the labeling as follows.

$$f(x_1) = 1; f(x_2) = n + 2; f(x_3) = 2; f(x_4) = n + 3; f(x_5) = n + 4; f(y_j) = 2n - j + 6;$$

$$f(x_1x_2) = n + 7; f(x_2x_3) = n + 6; f(x_3x_4) = n + 5; f(x_4x_5) = 3 \text{ and}$$

$$f(x_1y_j) = j + 3, \text{ for all } 1 \leq j \leq n - 2.$$

Because each edge holds $f(u) + f(uv) + f(v) = k$. So, it is proved that for $n \geq 2$, then the broom graph $SP_{3,n}$ has an edge-magic total labeling with $k = 2n + 10$.

2. Super Edge-Magic Total Labeling of a broom Graph $SP_{3,n}$

Theorem 3.8. If $n \geq 2$, then the broom graph $SP_{3,n}$ has a super edge-magic total labeling with $k = 3n + 7$.

Proof. Suppose the broom $SP_{3,n}$ obtains a set of vertices and a set of edges

$$V(SP_{3,n}) = \{x_1, x_2, x_3, x_4, x_5\} \cup \{y_j | 1 \leq j \leq n - 2\} \text{ and}$$

$$E(SP_{3,n}) = \{x_1x_2, x_2x_3, x_3x_4, x_4x_5\} \cup \{x_1y_j | 1 \leq j \leq n - 2\}.$$

The cardinality of the vertex set and edge set are respectively $|V(SP_{3,n})| = p = n + 3$ and $|E(SP_{3,n})| = q = n + 2$. Define a bijection f from $V(SP_{3,n}) \cup E(SP_{3,n})$ to $\{1, 2, \dots, p + q\}$ with the labeling as follows.

$$f(x_1) = n + 3; f(x_2) = 2; f(x_3) = n + 2; f(x_4) = 1; f(x_5) = n + 1; f(y_j) = j + 2;$$

$$f(x_1x_2) = 2n + 2; f(x_2x_3) = 2n + 3; f(x_3x_4) = 2n + 4; f(x_4x_5) = 2n + 5 \text{ and}$$

$$f(x_1y_j) = 2n - j + 2, \text{ for all } 1 \leq j \leq n - 2.$$

Because $f(V(G)) = \{1, 2, \dots, p\}$ and each edge holds $f(u) + f(uv) + f(v) = k$. So, it is proved that for $n \geq 2$, then the broom graph $SP_{3,n}$ has a super edge-magic total labeling with $k = 3n + 7$.

c. Edges-Magic Total Labelings of a Broom Graph $SP_{4,n}$

There are two types of the edges-magic total labeling on a broom graph $SP_{4,n}$ as follows:

1. Edge-Magic Total Labeling of a Broom Graph $SP_{4,n}$

Theorem 3.9. If $n \in \mathbb{N}$, then the broom graph $SP_{4,n}$ has an edge-magic total labeling with $k = 2n + 10$.

Proof. Suppose the broom $SP_{4,n}$ obtains a set of vertices and a set of edges

$$V(SP_{4,n}) = \{x_1, x_2, x_3, x_4\} \cup \{y_j | 1 \leq j \leq n - 1\} \text{ and}$$

$$E(SP_{4,n}) = \{x_1x_2, x_2x_3, x_3x_4\} \cup \{x_1y_j | 1 \leq j \leq n - 1\}.$$

The cardinality of the vertex set and edge set are respectively $|V(SP_{4,n})| = p = n + 3$ and $|E(SP_{4,n})| = q = n + 2$. Define a bijection f from $V(SP_{4,n}) \cup E(SP_{4,n})$ to $\{1, 2, \dots, p + q\}$ with the labeling as follows.

$$f(x_1) = 2; f(x_2) = 3; f(x_3) = n + 4; f(x_4) = 1; f(y_j) = j + 3; f(x_1x_2) = 2n + 5;$$

$$f(x_2x_3) = n + 3; f(x_3x_4) = n + 5 \text{ and } f(x_1y_j) = 2n - j + 5, \text{ for all } 1 \leq j \leq n - 1.$$

Because each edge holds $f(u) + f(uv) + f(v) = k$. So, it is proved that for $n \in \mathbb{N}$, then the broom graph $SP_{4,n}$ has an edge-magic total labeling with $k = 2n + 10$.

2. Super Edge-Magic Total Labeling of a Broom Graph $SP_{4,n}$

Theorem 3.10. If $n \in \mathbb{N}$, then the broom graph $SP_{4,n}$ has a super edge-magic total labeling with $k = 2n + 9$.

Proof. Suppose $SP_{4,n}$ obtains a set of vertices and a set of edges

$$V(SP_{4,n}) = \{x_1, x_2, x_3, x_4\} \cup \{y_j | 1 \leq j \leq n - 1\} \text{ and}$$

$$E(SP_{4,n}) = \{x_1x_2, x_2x_3, x_3x_4\} \cup \{x_1y_j | 1 \leq j \leq n - 1\}.$$

The cardinality of the vertex set and edge set are respectively $|V(SP_{4,n})| = p = n + 3$ and $|E(SP_{4,n})| = q = n + 2$. Define a bijection f from $V(SP_{4,n}) \cup E(SP_{4,n})$ to $\{1, 2, \dots, p + q\}$ with the labeling as follows.

$$f(x_1) = 2; f(x_2) = 4; f(x_3) = 1; f(x_4) = 3; f(y_j) = j + 4; f(x_1x_2) = 2n + 3;$$

$$f(x_2x_3) = 2n + 4; f(x_3x_4) = 2n + 5 \text{ and } f(x_1y_j) = 2n - j + 3, \text{ for all } 1 \leq j \leq n - 1.$$

Because $f(V(G)) = \{1, 2, \dots, p\}$ and each edge holds $f(u) + f(uv) + f(v) = k$. So, it is proved that for $n \in \mathbb{N}$, then the broom graph $SP_{4,n}$ has a super edge-magic total labeling with $k = 2n + 9$.

d. Edge-Magic Total Labeling of a Graph $Amal(S_n; P_4, x_1)$

There are two types of the edges-magic total labeling on a graph $Amal(S_n; P_4, x_1)$ as follows:

1. Edge-Magic Total Labeling of a Graph $Amal(S_n; P_4, x_1)$

Theorem 3.11. If $n \in \mathbb{N}$, then the graph $Amal(S_n; P_4, x_1)$ has an edge-magic total labeling with $k = 2n + 10$.

Proof. Suppose the graph $Amal(S_n; P_4, x_1)$ obtains a set of vertices and a set of edges

$$V(Amal(S_n; P_4, x_1)) = \{x_1, x_2, x_3\} \cup \{y_1, y_2, \dots, y_j\}, \text{ for } 1 \leq j \leq n \text{ and}$$

$$E(Amal(S_n; P_4, x_1)) = \{x_1x_2, x_2x_3\} \cup \{x_1y_1, x_1y_2, \dots, x_1y_j\}, \text{ for } 1 \leq j \leq n.$$

The cardinality of the vertex set and edge set are respectively $|V(Amal(S_n; P_4, x_1))| = p = n + 3$ and $|E(Amal(S_n; P_4, x_1))| = q = n + 2$. Define a bijection f from $V(Amal(S_n; P_4, x_1)) \cup E(Amal(S_n; P_4, x_1))$ to $\{1, 2, \dots, p + q\}$ with the labeling as follows.

$$f(x_1) = 1; f(x_2) = 2n + 5; f(x_3) = 2; f(y_j) = j + 4; f(x_1x_2) = 4; f(x_2x_3) = 3 \text{ and}$$

$$f(x_1y_j) = 2n - j + 5, \text{ for all } 1 \leq j \leq n.$$

Because each edge holds $f(u) + f(uv) + f(v) = k$. So, it is proved that for $n \in \mathbb{N}$, then the graph $Amal(S_n; P_4, x_1)$ has an edge-magic total labeling with $k = 2n + 10$.

2. Super Edge-Magic Total Labeling of a Graph $Amal(S_n; P_4, x_1)$

Theorem 3.12. If $n \in \mathbb{N}$, then the graph $Amal(S_n; P_4, x_1)$ has a super edge-magic total labeling with $k = 2n + 9$.

Proof. Suppose the graph $Amal(S_n; P_4, x_1)$ obtains a set of vertices and a set of edges

$$V(Amal(S_n; P_4, x_1)) = \{x_1, x_2, x_3\} \cup \{y_j | 1 \leq j \leq n\} \text{ and}$$

$$E(Amal(S_n; P_4, x_1)) = \{x_1x_2, x_2x_3\} \cup \{x_1y_j | 1 \leq j \leq n\}.$$

The cardinality of the vertex set and edge set are respectively $|V(Amal(S_n; P_4, x_1))| = p = n + 3$ and $|E(Amal(S_n; P_4, x_1))| = q = n + 2$. Define a bijection f from $V(Amal(S_n; P_4, x_1)) \cup E(Amal(S_n; P_4, x_1))$ to $\{1, 2, \dots, p + q\}$ with the labeling as follows.

$$f(x_1) = 2; f(x_2) = 3; f(x_3) = 1; f(y_j) = j + 3; f(x_1x_2) = 2n + 4; f(x_2x_3) = 2n + 5 \text{ and}$$

$$f(x_1y_j) = 2n - j + 4, \text{ for all } 1 \leq j \leq n.$$

Because $f(V(G)) = \{1, 2, \dots, p\}$ and each edge holds $f(u) + f(uv) + f(v) = k$. So, it is proved that for $n \in \mathbb{N}$, then the graph $Amal(S_n; P_4, x_1)$ has a super edge-magic total labeling with $k = 2n + 9$.

CONCLUSION

In this paper, we have given some solutions, to solve the edge-magic total and super edge-magic total labeling including intervals of the magic sums on vertex amalgamation graphs of a star graph S_n of order n with a path graph P_4 . However, it is very difficult to solve the labeling problem using traditional manual problem. Finally, we conclude that manual method to solve the labeling problem in this case proved to be totally effective. The authors would like to thank to the reviewers for helping comments and suggestions, which have improved the presentation. I personally would also like to show my deep appreciation to my supervisors who helped me finalize our study.

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